Linear Beamforming for the Spatially Correlated MISO Broadcast Channel

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Abstract—A spatially correlated broadcast setting with M antennas at the base station and M users (each with a single antenna) is considered. We assume that the users have perfect channel information about their links and the base station has only statistical information about each user's link. The base station employs a linear beamforming strategy with one spatial eigen-mode allocated to each user. The goal of this work is to understand the structure of the beamforming vectors that maximize the ergodic sum-rate achieved by treating interference as noise. In the $M = 2$ case, we first fix the beamforming vectors and compute the ergodic sum-rate in closed-form as a function of the channel statistics. We then show that the optimal beamforming vectors are the dominant generalized eigenvectors of the covariance matrices of the two links. It is difficult to obtain intuition on the structure of the optimal beamforming vectors for $M > 2$ due to the complicated nature of the sum-rate expression. Nevertheless, in the case of asymptotic M, we show that the optimal beamforming vectors have to satisfy a set of fixed-point equations.

I. INTRODUCTION

The focus of this paper is on a MISO broadcast (downlink) setting where the base station (BS) has M antennas with M users in the cell, each having a single antenna. Under the assumption of perfect channel state information (CSI) at both the ends, significant progress has been made over the last few years on understanding optimal signaling that achieves the sum-capacity [1]–[5] as well as the capacity region [6] of the multi-antenna broadcast channel. Though the capacity-achieving *dirty paper coding* scheme is well-understood, the complexity associated with it makes it an impractical choice. Thus, recent focus has been on a family of linear precoding schemes [7]–[9] which are within a fixed power-offset of the dirty paper coding scheme. In particular, a linear beamforming scheme that allocates one eigen-mode to each user is of considerable interest in standardization efforts.

More importantly, while reasonably accurate CSI can be obtained at the users via pilot-based schemes, CSI at the BS requires either channel reciprocity or reverse link feedback, both of which put an overwhelming burden on the operating cost. Thus, there has been a significant interest on understanding the information-theoretic limits of broadcast channels under practical assumptions on CSI. In the extreme case of no CSI at the BS, the multiplexing gain possible in the perfect CSI case (M) is lost completely as it reduces to one.

The no CSI assumption is pessimistic and in practice, the channel evolves fairly slowly on a statistical scale and it is possible to learn the statistics of the individual links at the BS with minimal cost. In the MISO broadcast setting with a Rayleigh fading model for each user (zero mean complex Gaussian fading process), the complete channel statistics are specified by the covariance matrix of the vector channel of the user. In this context, it must be noted that initial works assume that all the users experience fading that is *independent and identically distributed* (i.i.d.) across the antennas. That is, the covariance matrix of each user is the identity matrix. This

assumption cannot be justified in practice unless the antennas at the BS are spaced wide apart and the scattering environment connecting the BS with the users is rich. While the correlated case has been studied in the literature [10], [11], the general version of the problem studied here has not received much attention.

The focus of this work is on understanding the impact of the users' spatial statistics (their covariance matrices) on the sum-rate performance of the linear beamforming scheme. We first study the simplest non-trivial case of $M = 2$ and compute the sum-rate achievable with a linear beamforming scheme under the practical assumption that interference is treated as noise. For this, we exploit knowledge of the structure of density function of the weighted norm of isotropically distributed beamforming vectors [12]. Our sum-rate characterization is explicit and in terms of the covariance matrices of the two users and the beamforming vectors.

While identifying the structure of the sum-rate optimizing beamforming vectors is a difficult problem, in general, we obtain intuition in the low- and the high-SNR extremes. In the low-SNR extreme, it is not surprising that a strategy where the BS beamforms along the dominant eigen-mode of each user's channel is sum-rate optimal. In the high-SNR extreme, a strategy where the BS beamforms to a given user along the dominant generalized eigenvector[1] of that user's and the other user's covariance matrices is sum-rate optimal. Intuitively speaking, given that the BS has only statistical information of the two links, it generates an "effective" covariance matrix for a particular user by statistically pre-nulling the interference from the forward channel of the user. The sum-rate optimal beamforming vectors are the dominant eigen-modes of these effective covariance matrices. Solutions in terms of the generalized eigenvectors are obtained in the perfect CSI case [9], [13], but to the best of our knowledge, this solution in the statistical case is a first. While the generalization of this result to the $M > 2$ case is cumbersome, simple approximations for the ergodic sum-rate in terms of the channel statistics are provided in the asymptotics of M. Based on these approximations, we show that the optimal beamforming vectors are solutions to a set of fixed-point equations.

Note: Due to space constraints, the proofs of the main statements in this paper are not provided and the logic of the main arguments are sketched out in brief.

II. SYSTEM SETUP

We consider a broadcast setting with M antennas at the base station (BS) and M users, each with a single antenna. We denote the $M \times 1$ channel between the BS and user i as

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[1] A generalized eigenvector $\mathbf{x}$ (with the corresponding generalized eigenvalue σ) of a pair of matrices $(\mathbf{A}, \mathbf{B})$ satisfies the relationship $\mathbf{A}\,\mathbf{x} = \sigma\,\mathbf{B}\,\mathbf{x}$. In the special case where $\mathbf{B}$ is invertible, a generalized eigenvector of the pair $(\mathbf{A}, \mathbf{B})$ is an eigenvector of $\mathbf{B}^{-1}\mathbf{A}$. If $\mathbf{A}$ and $\mathbf{B}$ are also positive definite, then all the generalized eigenvalues are also positive.

$\mathbf{h}_i$, $i = 1, \cdots, M$. While different multi-user communication strategies can be considered, as motivated in the Introduction, the focus here is on a linear beamforming scheme where the BS beamforms the information-bearing signal s_i meant for user i with the $M \times 1$ unit-normed vector $\mathbf{w}_i$. We assume that s_i is unit energy and the BS divides its power budget of ρ equally across all the users. The received symbol y_i at user i is written as

$$ y_i = \sqrt{\frac{\rho}{M}} \cdot \mathbf{h}_i^H \left(\sum_{i=1}^{M} \mathbf{w}_i s_i \right) + n_i, \ i = 1, \cdots, M $$

where n_i denotes the $\mathcal{CN}(0, 1)$ complex Gaussian noise added at the receiver.

We assume a Rayleigh fading (zero mean complex Gaussian) model for the channel and hence, the complete spatial statistics are described by the second-order moments of $\{\mathbf{h}_i\}$. With M antennas at the BS and a single antenna at each user, the channel $\mathbf{h}_i$ of user i can be generically written as

$$ \mathbf{h}_i = \boldsymbol{\Sigma}_i^{1/2} \mathbf{h}_{\text{iid}, i} \tag{1} $$

where $\mathbf{h}_{\text{iid}, i}$ is an $M \times 1$ vector with i.i.d. $\mathcal{CN}(0,1)$ entries and $\boldsymbol{\Sigma}_i$ is the covariance matrix corresponding to the user i. In particular, with $\boldsymbol{\Sigma}_i = \mathbf{I}_M$ for all users, (1) reduces to the i.i.d. downlink model well-studied in the literature.

The metric of interest in this work is the throughput from the BS to the users. Under the assumption of Gaussian inputs $\{s_i\}$, the instantaneous information-theoretic[2] rate, R_i, achievable by user i with the linear beamforming scheme and using a mismatched decoder is given by

$$ R_i = \log \left(1 + \frac{\frac{\rho}{M} \cdot |\mathbf{h}_i^H \mathbf{w}_i|^2}{1 + \frac{\rho}{M} \cdot \sum_{j \neq i} |\mathbf{h}_i^H \mathbf{w}_j|^2} \right) $$

$$ = \underbrace{\log \left(1 + \frac{\rho}{M} \sum_{j=1}^{M} |\mathbf{h}_i^H \mathbf{w}_j|^2 \right)}_{I_{i,1}} - \underbrace{\log \left(1 + \frac{\rho}{M} \sum_{j \neq i} |\mathbf{h}_i^H \mathbf{w}_j|^2 \right)}_{I_{i,2}}. $$

In particular, the ergodic sum-rate achievable with the linear beamforming scheme is given by

$$ \mathcal{R} \triangleq \sum_{i=1}^{M} E[R_i]. $$

With the spatial correlation model assumed in (1), we can write $I_{i,1}$ as

$$ I_{i,1} = \log \left(1 + \frac{\rho}{M} \cdot \mathbf{h}_{\text{iid}, i}^H \boldsymbol{\Sigma}_i^{1/2} \left(\sum_{j=1}^{M} \mathbf{w}_j \mathbf{w}_j^H \right) \boldsymbol{\Sigma}_i^{1/2} \mathbf{h}_{\text{iid}, i} \right) $$

$$ = \log \left(1 + \frac{\rho}{M} \cdot \mathbf{h}_{\text{iid}, i}^H \mathbf{V}_i \boldsymbol{\Lambda}_i \mathbf{V}_i^H \mathbf{h}_{\text{iid}, i} \right), $$

where we have used the following eigen-decomposition in the second equation:

$$ \mathbf{V}_i \boldsymbol{\Lambda}_i \mathbf{V}_i^H = \boldsymbol{\Sigma}_i^{1/2} \left(\sum_{j=1}^{M} \mathbf{w}_j \mathbf{w}_j^H \right) \boldsymbol{\Sigma}_i^{1/2} $$

$$ \boldsymbol{\Lambda}_i = \mathsf{diag}\left([\boldsymbol{\Lambda}_{i,1}, \cdots, \boldsymbol{\Lambda}_{i,M}] \right), \ \boldsymbol{\Lambda}_{i,1} \geq \cdots \geq \boldsymbol{\Lambda}_{i,M} \geq 0. $$

Similarly, we can write $I_{i,2}$ as

$$ I_{i,2} = \log \left(1 + \frac{\rho}{M} \cdot \mathbf{h}_{\text{iid}, i}^H \widetilde{\mathbf{V}}_i \widetilde{\boldsymbol{\Lambda}}_i \widetilde{\mathbf{V}}_i^H \mathbf{h}_{\text{iid}, i} \right) $$

$$ \widetilde{\mathbf{V}}_i \widetilde{\boldsymbol{\Lambda}}_i \widetilde{\mathbf{V}}_i^H = \boldsymbol{\Sigma}_i^{1/2} \left(\sum_{j \neq i} \mathbf{w}_j \mathbf{w}_j^H \right) \boldsymbol{\Sigma}_i^{1/2} $$

$$ \widetilde{\boldsymbol{\Lambda}}_i = \mathsf{diag}\left([\widetilde{\boldsymbol{\Lambda}}_{i,1}, \cdots, \widetilde{\boldsymbol{\Lambda}}_{i,M}] \right), \ \widetilde{\boldsymbol{\Lambda}}_{i,1} \geq \cdots \geq \widetilde{\boldsymbol{\Lambda}}_{i,M} \geq 0. $$

Towards the goal of computing the ergodic rates, we expand $\mathbf{h}_{\text{iid}, i}$ into its magnitude and directional components as $\mathbf{h}_{\text{iid}, i} = \|\mathbf{h}_{\text{iid}, i}\| \cdot \widetilde{\mathbf{h}}_{\text{iid}, i}$. Note that $\|\mathbf{h}_{\text{iid}, i}\|^2$ can be written as

$$ \|\mathbf{h}_{\text{iid}, i}\|^2 = \frac{1}{2} \sum_{j=1}^{2M} z_j^2 $$

where z_j^2 is a standard (real) chi-squared random variable and $\widetilde{\mathbf{h}}_{\text{iid}, i}$ is a unit-normed vector that is isotropically distributed on the surface of M-dimensional complex sphere. Thus, we can rewrite $I_{i,1}$ and $I_{i,2}$ as

$$ I_{i,1} = \log \left(1 + \frac{\rho}{M} \cdot \|\mathbf{h}_{\text{iid}, i}\|^2 \cdot \widetilde{\mathbf{h}}_{\text{iid}, i}^H \mathbf{V}_i \boldsymbol{\Lambda}_i \mathbf{V}_i^H \widetilde{\mathbf{h}}_{\text{iid}, i} \right) $$

$$ I_{i,2} = \log \left(1 + \frac{\rho}{M} \cdot \|\mathbf{h}_{\text{iid}, i}\|^2 \cdot \widetilde{\mathbf{h}}_{\text{iid}, i}^H \widetilde{\mathbf{V}}_i \widetilde{\boldsymbol{\Lambda}}_i \widetilde{\mathbf{V}}_i^H \widetilde{\mathbf{h}}_{\text{iid}, i} \right). $$

Further, since the magnitude and directional information of an i.i.d. (isotropically distributed) random vector are independent, $E[I_{i,1}]$ and $E[I_{i,2}]$ can be written as

$$ E[I_{i,1}] = E\left[\log \left(1 + \frac{\rho}{M} \cdot \|\mathbf{h}_{\text{iid}, i}\|^2 \cdot \widetilde{\mathbf{h}}_{\text{iid}, i}^H \boldsymbol{\Lambda}_i \widetilde{\mathbf{h}}_{\text{iid}, i} \right) \right] $$

$$ E[I_{i,2}] = E\left[\log \left(1 + \frac{\rho}{M} \cdot \|\mathbf{h}_{\text{iid}, i}\|^2 \cdot \widetilde{\mathbf{h}}_{\text{iid}, i}^H \widetilde{\boldsymbol{\Lambda}}_i \widetilde{\mathbf{h}}_{\text{iid}, i} \right) \right] $$

where we have also used the fact that a fixed[3] unitary transformation of an isotropically distributed vector on the surface of the complex sphere does not alter its distribution.

III. ERGODIC SUM-RATE: TWO USER CASE

The focus of this section is on computing the ergodic information-theoretic rates in closed-form in the special case of two users ($M = 2$). This closed-form expression will be a function of the covariance matrices of the two users, $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$, and the choice of beamforming vectors, $\mathbf{w}_1$ and $\mathbf{w}_2$. Once a closed-form expression is obtained, our goal lies in characterizing the structure of the optimal beamforming vectors as a function of the channel statistics and SNR.

For simplicity, we assume that

$$ \boldsymbol{\Sigma}_1 = \mathbf{U} \, \mathsf{diag}([\lambda_1 \ \lambda_2]) \, \mathbf{U}^H, \ \boldsymbol{\Sigma}_2 = \widetilde{\mathbf{U}} \, \mathsf{diag}([\mu_1 \ \mu_2]) \, \widetilde{\mathbf{U}}^H \tag{2} $$

where $\mathbf{U} = [\mathbf{u}_1(\boldsymbol{\Sigma}_1), \ \mathbf{u}_2(\boldsymbol{\Sigma}_1)]$, $\widetilde{\mathbf{U}} = [\mathbf{u}_1(\boldsymbol{\Sigma}_2), \ \mathbf{u}_2(\boldsymbol{\Sigma}_2)]$, $\lambda_1 \geq \lambda_2 > 0$ and $\mu_1 \geq \mu_2 > 0$ (that is, both $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$ are positive definite). Define the condition numbers κ_1 and κ_2 as

$$ \kappa_1 \triangleq \frac{\lambda_1}{\lambda_2} \ \text{and} \ \kappa_2 \triangleq \frac{\mu_1}{\mu_2}. \tag{3} $$

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[3]Note that the unitary transformation is independent of the channel realization when the beamforming vectors are chosen based on long-term statistics of the channel.

[2]All rate quantities will be assumed to be in nats/s/Hz in this work.

Proposition 1: The ergodic information-theoretic rate achievable at user i (where $i = 1, 2$) with linear beamforming in the two user case is given by

$$E[R_i] = E[I_{i,1}] - E[I_{i,2}]$$
$$= \frac{\mathbf{\Lambda}_{i,1} \cdot e^{\frac{2}{\rho \mathbf{\Lambda}_{i,1}}} E_1\left(\frac{2}{\rho \mathbf{\Lambda}_{i,1}}\right) - \mathbf{\Lambda}_{i,2} \cdot e^{\frac{2}{\rho \mathbf{\Lambda}_{i,2}}} E_1\left(\frac{2}{\rho \mathbf{\Lambda}_{i,2}}\right)}{\mathbf{\Lambda}_{i,1} - \mathbf{\Lambda}_{i,2}}$$
$$- \exp\left(2/\rho \widetilde{\mathbf{\Lambda}}_{i,1}\right) E_1\left(2/\rho \widetilde{\mathbf{\Lambda}}_{i,1}\right)$$

where $E_1(x) = \int_x^\infty \frac{e^{-t}}{t} dt$ is the exponential integral. The corresponding eigenvalues can be written in terms of $\mathbf{\Sigma}_i$ and the beamforming vectors as follows:

$$\mathbf{\Lambda}_{i,1} = \frac{A_i + B_i + \sqrt{(A_i - B_i)^2 + 4C_i^2}}{2}$$
$$\mathbf{\Lambda}_{i,2} = \frac{A_i + B_i - \sqrt{(A_i - B_i)^2 + 4C_i^2}}{2}$$
$$\widetilde{\mathbf{\Lambda}}_{i,1} = B_i$$
$$\widetilde{\mathbf{\Lambda}}_{i,2} = 0$$

where $A_i = \mathbf{w}_i^H \mathbf{\Sigma}_i \mathbf{w}_i$, $B_i = \mathbf{w}_j^H \mathbf{\Sigma}_i \mathbf{w}_j$ and $C_i = |\mathbf{w}_i^H \mathbf{\Sigma}_i \mathbf{w}_j|$ with $j \neq i$ and $\{i, j\} = 1, 2$.

Proof: Note that a closed-form computation of $E[I_{i,1}]$ requires the density function of weighted norms of isotropically distributed unit-normed vectors since

$$E[I_{i,1}] = E_{\mathbf{X}}\left[\int_{y=\mathbf{\Lambda}_{i,2}}^{\mathbf{\Lambda}_{i,1}} \log(1 + Xy) \, \mathsf{P}_i(y) dy\right]$$

where $\mathbf{X}$ denotes the random variable $\mathbf{X} = \frac{\rho}{2} \cdot \|\mathbf{h}_{\mathsf{iid},i}\|^2$ and X corresponds to a realization of $\mathbf{X}$. Let $\mathbf{Y}$ denote the random variable

$$\mathbf{Y} \triangleq \widetilde{\mathbf{h}}_{\mathsf{iid},i}^H \mathbf{\Lambda}_i \widetilde{\mathbf{h}}_{\mathsf{iid},i} = \sum_{j=1}^2 \mathbf{\Lambda}_{i,j} \left|\widetilde{\mathbf{h}}_{\mathsf{iid},i}(j)\right|^2.$$

The Ritz-Rayleigh relationship implies that $\mathbf{\Lambda}_{i,2} \leq \mathbf{Y} \leq \mathbf{\Lambda}_{i,1}$ and the density function of $\mathbf{Y}$ evaluated at y is denoted as $\mathsf{P}_i(y)$. In [12], $\mathsf{P}_i(y)$ in the $M = 2$ case is shown to be uniform, that is,

$$\mathsf{P}_i(y) = \frac{1}{\mathbf{\Lambda}_{i,1} - \mathbf{\Lambda}_{i,2}}, \quad \mathbf{\Lambda}_{i,2} \leq y \leq \mathbf{\Lambda}_{i,1}.$$

The statement of the proposition follows from a routine computation via the integral tables [14]. ∎

Note that understanding the structure of the optimal choice of beamforming vectors, $(\mathbf{w}_{1,\mathsf{opt}}, \mathbf{w}_{2,\mathsf{opt}})$, that maximize the ergodic sum-rate as a function of $\mathbf{\Sigma}_1$, $\mathbf{\Sigma}_2$ and ρ is a hard problem, in general. Therefore, we consider the low- and the high-SNR extremes to obtain insights.

*Low-*SNR *Extreme:* We need the following characterization of the exponential integral:

$$\frac{1}{x+2} \leq \frac{1}{2}\log\left(1 + \frac{2}{x}\right) \leq E_1(x)e^x \leq \log\left(1 + \frac{1}{x}\right) \leq \frac{1}{x}$$

where the extremal inequalities are established by using the fact that $\frac{x}{x+1} \leq \log(1+x) \leq x$. Note that the upper and the lower bounds get tight as $x \to \infty$ (or $\rho \to 0$ in this context). Using the above bound, we have

$$E[R_i] \stackrel{\rho \to 0}{\to} \frac{\rho}{2}(\mathbf{\Lambda}_{i,1} + \mathbf{\Lambda}_{i,2} - B_i) = \frac{\rho}{2} \cdot A_i = \frac{\rho}{2} \cdot \mathbf{w}_i^H \mathbf{\Sigma}_i \mathbf{w}_i.$$

In the low-SNR regime, the system is noise-limited and hence, the linear scaling of $E[R_i]$ with SNR. It is also straightforward to note that maximizing $E[R_i]$ is contingent on optimizing over $\mathbf{w}_i$ alone. Thus, the sum-rate is maximized by

$$\mathbf{w}_{1,\mathsf{opt}} = \mathbf{u}_1(\mathbf{\Sigma}_1) \quad \text{and} \quad \mathbf{w}_{2,\mathsf{opt}} = \mathbf{u}_1(\mathbf{\Sigma}_2)$$

where $\mathbf{u}_1(\mathbf{\Sigma}_i)$ denotes the dominant eigenvector (an eigenvector corresponding to the dominant eigenvalue) of $\mathbf{\Sigma}_i$. In other words, in the low-SNR extreme, each user signals along the optimal statistical eigen-mode of its channel (and ignoring the other user's channel completely). This conclusion should not be entirely surprising. The resulting ergodic sum-rate is given as

$$\mathcal{R} \stackrel{\rho \to 0}{\to} \frac{\rho}{2} \cdot \left[\lambda_{\max}(\mathbf{\Sigma}_1) + \lambda_{\max}(\mathbf{\Sigma}_2)\right].$$

High-SNR ***Extreme:*** The following expansion of the exponential integral is useful in characterizing $\mathcal{R}$ as $\rho \to \infty$:

$$E_1(x) = \log\left(\frac{1}{x}\right) + \sum_{k=1}^\infty \frac{(-1)^{k+1} x^k}{k \cdot k!} - \gamma$$
$$\stackrel{x \to 0}{\to} \log\left(\frac{1}{x}\right) + x - \gamma$$

where $\gamma \approx 0.577$ is the Euler-Mascheroni constant. Using the above approximation, we have

$$E[R_i] \stackrel{\rho \to \infty}{\to} \frac{\mathbf{\Lambda}_{i,1}\log(\mathbf{\Lambda}_{i,1}) - \mathbf{\Lambda}_{i,2}\log(\mathbf{\Lambda}_{i,2})}{\mathbf{\Lambda}_{i,1} - \mathbf{\Lambda}_{i,2}} - \log(B_i).$$

The dominating impact of interference (due to the fixed nature of the linear beamforming scheme where the beamforming vectors are not adapted to the channel realizations) and the consequent boundedness of $E[R_i]$ as SNR increases should not be surprising. After some elementary manipulation, we can write $E[R_i]$ as

$$2E[R_i] \stackrel{\rho \to \infty}{\to} \log\left(\frac{A_i B_i - C_i^2}{B_i^2}\right) + \frac{A_i + B_i}{\sqrt{(A_i - B_i)^2 + 4C_i^2}} \cdot$$
$$\log\left(\frac{A_i + B_i + \sqrt{(A_i - B_i)^2 + 4C_i^2}}{A_i + B_i - \sqrt{(A_i - B_i)^2 + 4C_i^2}}\right)$$

We now rewrite the high-SNR ergodic rates in a form that eases further study.

Proposition 2: Define $d_{\mathbf{\Sigma}_i}(\mathbf{w}_1, \mathbf{w}_2)$ between two unit-normed vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ from the Grassmann manifold $\mathcal{G}(2, 1)$ as

$$d_{\mathbf{\Sigma}_i}(\mathbf{w}_1, \mathbf{w}_2) \triangleq \sqrt{\frac{4(A_i B_i - C_i^2)}{(A_i + B_i)^2}},$$

where A_i, B_i and C_i are as in the statement of Prop. 1.

- (a) Then, $d_{\mathbf{\Sigma}_i}(\mathbf{w}_1, \mathbf{w}_2)$ is a generalized "distance" semi-metric[4] between $\mathbf{w}_1$ and $\mathbf{w}_2$ satisfying $0 \leq d_{\mathbf{\Sigma}_i}(\cdot, \cdot) \leq 1$.
- (b) We can recast the ergodic rate in terms of $d_{\mathbf{\Sigma}_i}(\mathbf{w}_1, \mathbf{w}_2)$ as

$$E[R_i] + \log(2) \stackrel{\rho \to \infty}{\to} \frac{g(d_{\mathbf{\Sigma}_i}(\mathbf{w}_1, \mathbf{w}_2))}{2} + \log\left(1 + \frac{A_i}{B_i}\right)$$

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[4]A semi-metric satisfies all the properties necessary for a distance metric, except the triangle inequality.

where

$$g(z) = f(z) + 2\log(z),$$
$$f(z) = \frac{1}{\sqrt{1-z^2}}\log\left(\frac{1+\sqrt{1-z^2}}{1-\sqrt{1-z^2}}\right).$$

- (c) While $f(\bullet)$ is monotonically decreasing as a function of its argument, $g(\bullet)$ is increasing with

$$2\log(2) = \lim_{z\to 0} g(z) \leq \quad g(z) \quad \leq \lim_{z\to 1} g(z) = 2$$
$$\infty = \lim_{z\to 0} f(z) \geq \quad f(z) \quad \geq \lim_{z\to 1} f(z) = 2.$$

∎

We are now prepared to illustrate the structure of the optimal beamforming vectors.

Theorem 1: The optimal choice of the pair $(\mathbf{w}_{1,\,\text{opt}}, \mathbf{w}_{2,\,\text{opt}})$ that maximizes $E[R_i]$ in the high-SNR regime is

$$\mathbf{w}_{i,\,\text{opt}} = e^{j\nu_1}\mathbf{u}_1(\mathbf{\Sigma}_i) \quad \text{and} \quad \mathbf{w}_{j,\,\text{opt}} = e^{j\nu_2}\mathbf{u}_2(\mathbf{\Sigma}_i),\ j\neq i$$

for some choice of $\nu_i \in [0, 2\pi)$, $i = 1, 2$.

Proof: Let $\chi(\mathbf{\Sigma}_i) = \frac{\lambda_{\max}(\mathbf{\Sigma}_i)}{\lambda_{\min}(\mathbf{\Sigma}_i)}$ denote the condition number of $\mathbf{\Sigma}_i$. We first note that the optimization problem over the choice of a pair $(\mathbf{w}_1, \mathbf{w}_2)$ that results in a corresponding choice of (A_i, B_i, C_i) can be recast in the form of a two parameter optimization problem over (M_i, N_i) with $M_i = \frac{A_i}{B_i}$, $N_i = \frac{C_i}{B_i}$ under the constraint that $0 \leq N_i^2 \leq M_i \leq \chi(\mathbf{\Sigma}_i)$. This results in the following high-SNR expression:

$$2E[R_i] + 2\log(2) = g\left(\frac{2\sqrt{M_i - N_i^2}}{M_i + 1}\right) + 2\log(1 + M_i).$$

It is straightforward to show that the choice in the theorem maximizes the above equation. ∎

With this choice of beamforming vectors, $d_{\mathbf{\Sigma}_i}(\cdot, \cdot)$ and $E[R_i]$ can be written as

$$d_{\mathbf{\Sigma}_i}(\mathbf{w}_{i,\,\text{opt}}, \mathbf{w}_{j,\,\text{opt}}) = \frac{2\sqrt{\kappa_i}}{\kappa_i + 1}$$
$$E[R_i] \stackrel{\rho\to\infty}{\to} \frac{\kappa_i \log(\kappa_i)}{\kappa_i - 1},$$

whereas $\lim_{\rho\to\infty} E[R_j]$ is dependent on how the eigenvectors of $\mathbf{\Sigma}_i$ are related to $\mathbf{\Sigma}_j$. It is also to be noted that $E[R_i]$ increases (and $d_{\mathbf{\Sigma}_i}(\cdot, \cdot)$ decreases) as κ_i increases. That is, the more ill-conditioned $\mathbf{\Sigma}_i$ is, the larger the high-SNR statistical beamforming rate asymptote and *vice versa*. This should be intuitive as our goal is only to maximize $E[R_i]$ and the above choice achieves that goal.

We now consider the sum-rate setting restricted to the case where $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ have the same set of orthonormal eigenvectors. Instead of using the definitions of $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ as in (2), for simplicity, we will assume that $\mathbf{U} = \widetilde{\mathbf{U}} = [\mathbf{u}_1, \mathbf{u}_2]$. We define κ_1 and κ_2 as in (3). Without loss in generality, we can also assume that $\kappa_1 > 1$. Three possibilities arise depending on the relationship between 1, κ_1 and κ_2: i) $\kappa_1 > 1 \geq \kappa_2$, ii) $\kappa_1 > \kappa_2 > 1$, and iii) $\kappa_2 \geq \kappa_1 > 1$. (Note that first case subsumes the setting where $\mu_1 = \mu_2 = \mu$ and $\mathbf{\Sigma}_2 = \mu\mathbf{I}$.) The main result is the following theorem.

Theorem 2: The sum-rate is maximized by the following choice of beamforming vectors:

$$\mathbf{w}_{1,\,\text{opt}} = e^{j\nu_1}\mathbf{u}_1, \quad \mathbf{w}_{2,\,\text{opt}} = e^{j\nu_2}\mathbf{u}_2 \quad \text{if i) or ii) is true,}$$
$$\mathbf{w}_{1,\,\text{opt}} = e^{j\nu_2}\mathbf{u}_2, \quad \mathbf{w}_{2,\,\text{opt}} = e^{j\nu_1}\mathbf{u}_1 \quad \text{if iii) is true}$$

for some choice of $\nu_i \in [0, 2\pi)$, $i = 1, 2$. The optimal sum-rate is given as

$$E[R_1] + E[R_2] \stackrel{\rho\to\infty}{\to} \begin{cases} \frac{\kappa_1 \cdot \log(\kappa_1)}{\kappa_1 - 1} + \frac{\log(\kappa_2)}{\kappa_2 - 1} & \text{if } \kappa_1 \geq \kappa_2 \\ \frac{\kappa_2 \cdot \log(\kappa_2)}{\kappa_2 - 1} + \frac{\log(\kappa_1)}{\kappa_1 - 1} & \text{if } \kappa_1 < \kappa_2 \end{cases}$$

Proof: The proof follows by decomposing $\mathbf{w}_1$ and $\mathbf{w}_2$ along the obvious orthogonal basis of $\{\mathbf{u}_1, \mathbf{u}_2\}$:

$$\mathbf{w}_1 = \alpha\mathbf{u}_1 + \beta\mathbf{u}_2, \quad \mathbf{w}_2 = \gamma\mathbf{u}_1 + \delta\mathbf{u}_2$$

for some choice of $\{\alpha, \beta, \gamma, \delta\}$ with $\alpha = |\alpha|e^{j\theta_\alpha}$ (similarly, for other quantities) satisfying $|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\delta|^2 = 1$. A direct optimization of the high-SNR sum-rate expression shows that $\{\theta_\bullet\}$ enters the optimization only via the term $|\beta\gamma - \alpha\delta|$, which can be maximized by setting $\theta_\alpha + \theta_\delta - \theta_\beta - \theta_\gamma = \pi$ (modulo 2π). Parameterizing $|\alpha|$ and $|\gamma|$ as $|\alpha| = \sin(\theta)$ and $|\gamma| = \sin(\phi)$ for some $\{\theta, \phi\} \in [0, \pi/2]$, we can show that the sum-rate is maximized by $\theta = \pi/2$ and $\phi = 0$ if i) or ii) is true and by $\theta = 0$ and $\phi = \pi/2$ if iii) is true. For this, we establish an upper bound to the sum-rate and show that this bound is achieved by the choice as in the statement of the theorem. ∎

We now consider the general case where $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ do not have the same set of eigenvectors.

Theorem 3: In the general case, the sum-rate is maximized

$$\mathbf{w}_{1,\,\text{opt}} = e^{j\nu_1}\mathbf{u}_1(\mathbf{\Sigma}_2^{-1}\mathbf{\Sigma}_1), \quad \mathbf{w}_{2,\,\text{opt}} = e^{j\nu_2}\mathbf{u}_1(\mathbf{\Sigma}_1^{-1}\mathbf{\Sigma}_2)$$

for some choice of $\nu_i \in [0, 2\pi)$, $i = 1, 2$.

Proof: For this case, we define $\mathbf{\Sigma}$ and its corresponding eigen-decomposition as

$$\mathbf{\Sigma} \triangleq \mathbf{\Sigma}_2^{-\frac{1}{2}}\mathbf{\Sigma}_1\mathbf{\Sigma}_2^{-\frac{1}{2}} = \mathbf{V}\,\text{diag}\,([\eta_1\ \eta_2])\,\mathbf{V}^H$$

where $\mathbf{V} = [\mathbf{v}_1\ \mathbf{v}_2]$ and $\eta_1 \geq \eta_2$. Since $\mathbf{\Sigma}_2$ is a full rank matrix and $\{\mathbf{v}_1, \mathbf{v}_2\}$ form a basis, the vectors $\mathbf{\Sigma}_2^{-\frac{1}{2}}\mathbf{v}_1$ and $\mathbf{\Sigma}_2^{-\frac{1}{2}}\mathbf{v}_2$ also form a basis (albeit non-orthogonal, in general). We can decompose $\mathbf{w}_1$ and $\mathbf{w}_2$ along these vectors as

$$\mathbf{w}_1 = \frac{\alpha\mathbf{\Sigma}_2^{-\frac{1}{2}}\mathbf{v}_1 + \beta\mathbf{\Sigma}_2^{-\frac{1}{2}}\mathbf{v}_2}{\|\alpha\mathbf{\Sigma}_2^{-\frac{1}{2}}\mathbf{v}_1 + \beta\mathbf{\Sigma}_2^{-\frac{1}{2}}\mathbf{v}_2\|}, \quad \mathbf{w}_2 = \frac{\gamma\mathbf{\Sigma}_2^{-\frac{1}{2}}\mathbf{v}_1 + \delta\mathbf{\Sigma}_2^{-\frac{1}{2}}\mathbf{v}_2}{\|\gamma\mathbf{\Sigma}_2^{-\frac{1}{2}}\mathbf{v}_1 + \delta\mathbf{\Sigma}_2^{-\frac{1}{2}}\mathbf{v}_2\|}$$

for some choice of $\{\alpha, \beta, \gamma, \delta\}$ with $\alpha = |\alpha|e^{j\theta_\alpha}$ (similarly, for other quantities) satisfying $|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\delta|^2 = 1$. A suitable coordinate transformation at this stage results in an optimization problem that is related to the special case of Theorem 2. After this transformation, the proof follows along the same logic as in Theorem 2. ∎

The reason for the peculiar choice of decomposition in the above proof (instead of decomposing the beamforming vectors along $\{\mathbf{v}_1, \mathbf{v}_2\}$) is that $\mathbf{\Sigma}_2^{-\frac{1}{2}}\mathbf{v}_i$, $i = 1, 2$ turn out to be the dominant generalized eigenvectors of the pairs $(\mathbf{\Sigma}_1, \mathbf{\Sigma}_2)$ and $(\mathbf{\Sigma}_2, \mathbf{\Sigma}_1)$, respectively. Recall from Footnote 1 the definition of a generalized eigenvector. For the above claim, note that

$$\mathbf{\Sigma}_2^{-1}\mathbf{\Sigma}_1 = \mathbf{\Sigma}^{-\frac{1}{2}}\left(\mathbf{V}\,\text{diag}\,([\eta_1\ \eta_2])\,\mathbf{V}^H\right)\mathbf{\Sigma}_2^{\frac{1}{2}} = \mathbf{M}\,\mathbf{D}\,\mathbf{M}^{-1}$$

$$\mathbf{\Sigma}_1^{-1}\mathbf{\Sigma}_2 = \left(\mathbf{\Sigma}_2^{-1}\mathbf{\Sigma}_1\right)^{-1} = \mathbf{M}\,\mathbf{D}^{-1}\,\mathbf{M}^{-1}$$

where $\mathbf{M} = \mathbf{\Sigma}_2^{-\frac{1}{2}}\mathbf{V}$ and $\mathbf{D} = \text{diag}\,([\eta_1\ \eta_2])$. Theorem 2 is indeed a special case of Theorem 3. For this, note that the dominant eigenvector of $\mathbf{\Sigma}_2^{-1}\mathbf{\Sigma}_1$ is $\mathbf{u}_1$ and $\mathbf{u}_2$ when $\kappa_1 > \kappa_2$ and $\kappa_2 < \kappa_1$, respectively.

IV. Ergodic Sum-Rate: General M Case

A recent advance [15], [16] allows a computation of the density function of weighted sum of standard central chi-squared terms (*generalized* chi-squared random variables). Alternate to the approach of Prop. 1, this approach allows closed-form expressions in the general M case. For example, if $\mathbf{\Lambda}_i(j)$, $j = 1, \cdots, M$ are distinct[5], we have

$$E\left[I_{i,\,1}\right] = \sum_{k=1}^{M} \prod_{j=1,\,j\neq k}^{M} \frac{\mathbf{\Lambda}_i(k)}{\mathbf{\Lambda}_i(k) - \mathbf{\Lambda}_i(j)} \cdot x_k \qquad (4)$$

$$x_k = \exp\left(\frac{\rho}{\mathbf{\Lambda}_i(k)M}\right) E_1\left(\frac{\rho}{\mathbf{\Lambda}_i(k)M}\right).$$

For $E\left[I_{i,\,2}\right]$, replace $\mathbf{\Lambda}_i$ by $\widetilde{\mathbf{\Lambda}}_i$. It can be checked that this expression matches with the expression in the $M = 2$ case.

Nevertheless, it is important to note that the formula above is in terms of the eigenvalue matrices $\{\mathbf{\Lambda}_i, \widetilde{\mathbf{\Lambda}}_i, i = 1, \cdots, M\}$, which become harder (and impossible for $M \geq 5$) to compute in closed-form as a function of the beamforming vectors and the covariance matrices as M increases. Approximation to the generalized chi-squared random variable by a Gamma distribution with matching first two moments can also be used to produce sum-rate approximations. However, these approximations are of similar complexity as the above formula. In contrast, we now provide asymptotic approximations to the sum-rate directly in terms of the relevant variables.

Proposition 3: For any fixed ρ, the ergodic information-theoretic rate achievable at user i (where $i = 1, \cdots, M$) converges as $M \to \infty$ to

$$E\left[R_i\right] \quad \to \quad \log\left(1 + \mathsf{SINR}_i\right) \triangleq \mathcal{R}_{i,\,\infty}$$
$$\mathsf{SINR}_i \quad = \quad \frac{\frac{\rho}{M} \cdot \mathbf{w}_i^H \mathbf{\Sigma}_i \mathbf{w}_i}{1 + \frac{\rho}{M} \cdot \sum_{j=1,\,j\neq i}^{M} \mathbf{w}_j^H \mathbf{\Sigma}_i \mathbf{w}_j} \triangleq \frac{\mathsf{S}_i}{\mathsf{I}_i}.$$

Proof: The proof follows along a law of large numbers-type argument, strengthened to convergence in mean via a suitable truncation technique. ∎

Proposition 4: Based on the above expression, we have the following conclusions that mirror the main results of Sec. III. i) We have the following bound for $\sum_{i=1}^{M} \mathcal{R}_{i,\,\infty}$:

$$1 - \frac{\rho}{M} \cdot \max_{i=1,\cdots,M} \sum_{j=1}^{M} \mathbf{w}_j^H \mathbf{\Sigma}_i \mathbf{w}_j \leq \frac{\sum_{i=1}^{M} \mathcal{R}_{i,\,\infty}}{\frac{\rho}{M} \cdot \sum_{i=1}^{M} \mathbf{w}_i^H \mathbf{\Sigma}_i \mathbf{w}_i} \leq 1.$$

Thus, the optimal beamforming vectors as $\rho \to 0$ are such that $\mathbf{w}_{i,\,\mathrm{opt}} = \mathbf{u}_1(\mathbf{\Sigma}_i)$, $i = 1, \cdots, M$. ii) For any ρ, we have

$$\mathcal{R}_{i,\,\infty} \leq \log\left(1 + \frac{\frac{\rho}{M} \cdot \lambda_1(\mathbf{\Sigma}_i)}{1 + \frac{\rho}{M} \cdot \sum_{j=2}^{M} \lambda_j(\mathbf{\Sigma}_i)}\right)$$

and $\mathcal{R}_{i,\,\infty}$ is maximized by $\mathbf{w}_{i,\,\mathrm{opt}} = \mathbf{u}_1(\mathbf{\Sigma}_i)$, and

$$\left\{\mathbf{w}_{j,\,\mathrm{opt}}, j = 1, \cdots, M, j \neq i\right\} = \left\{\mathbf{u}_j(\mathbf{\Sigma}_i), j = 2, \cdots, M\right\}.$$

iii) $\sum_{i=1}^{M} \mathcal{R}_{i,\,\infty}$ is optimized by the set of beamforming vectors that solve the following fixed-point equations:

$$\frac{\mathbf{\Sigma}_i \mathbf{w}_i}{\mathsf{I}_i \cdot (1 + \mathsf{SINR}_i)} - \sum_{j\neq i} \frac{\mathsf{SINR}_j \cdot \mathbf{\Sigma}_j \mathbf{w}_i}{\mathsf{I}_j \cdot (1 + \mathsf{SINR}_j)} = \mathbf{0}, \; i = 1, \cdots, M.$$

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[5]More complicated expressions can be obtained in case $\{\mathbf{\Lambda}_i(j)\}$ are not distinct.

V. Conclusion

We have studied statistics-based linear beamformer design for the MISO broadcast channel in this work. Based on a closed-form computation of the ergodic sum-rate in the $M = 2$ (two-user) case, we provide intuition on the structure of the optimal beamforming vectors that maximize the sum-rate in the low- and the high-SNR extremes. While further intuition on the small M case seems difficult, in the asymptotics of M, we are able to obtain intuition on the structure of the optimal beamforming vectors. The case of optimal statistical linear beamforming design has not received much attention in the literature and our work sets the course for a systematic and low-complexity limited feedback design in the broadcast setting, which is of considerable importance in the standardization efforts.

References

[1] G. Caire and S. Shamai, "On the achievable throughput of a multi-antenna Gaussian broadcast channel," *IEEE Trans. Inf. Theory*, vol. 49, no. 7, pp. 1691–1706, July 2003.

[2] P. Viswanath and D. N. C. Tse, "Sum capacity of the vector Gaussian broadcast channel and downlink-uplink duality," *IEEE Trans. Inf. Theory*, vol. 49, no. 8, pp. 1912–1921, Aug. 2003.

[3] S. Vishwanath, N. Jindal, and A. Goldsmith, "Duality, achievable rates and sum rate capacity of Gaussian MIMO broadcast channel," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2658–2668, Oct. 2003.

[4] N. Jindal, S. Vishwanath, and A. Goldsmith, "On the duality of Gaussian multiple-access and broadcast channels," *IEEE Trans. Inf. Theory*, vol. 50, no. 5, pp. 768–783, May 2004.

[5] W. Yu and J. M. Cioffi, "Sum capacity of Gaussian vector broadcast channels," *IEEE Trans. Inf. Theory*, vol. 50, no. 9, pp. 1875–1892, Sept. 2004.

[6] H. Weingarten, Y. Steinberg, and S. Shamai, "The capacity region of the Gaussian multiple-input multiple-output broadcast channel," *IEEE Trans. Inf. Theory*, vol. 52, no. 9, pp. 3936–3964, Sept. 2006.

[7] M. Schubert and H. Boche, "Solution of multiuser downlink beamforming problem with individual SINR constraint," *IEEE Trans. Veh. Tech.*, vol. 53, no. 1, pp. 18–28, Jan. 2004.

[8] C. Peel, B. Hochwald, and A. Swindlehurst, "Vector perturbation techniques for near-capacity multiantenna multi-user communication," *IEEE Trans. Commun.*, vol. 53, no. 1, pp. 195–202, Jan. 2005.

[9] A. Wiesel, Y. C. Eldar, and S. Shamai, "Zero forcing precoding and generalized inverses," *IEEE Trans. Sig. Proc.*, vol. 56, no. 9, pp. 4409–4418, Sept. 2008.

[10] T. Y. Al-Naffouri, M. Sharif, and B. Hassibi, "How much does transmit correlation affect the sum-rate scaling of MIMO Gaussian broadcast channels?," *IEEE Trans. Commun.*, vol. 57, no. 2, pp. 562–572, Feb. 2009.

[11] M. Trivellato, F. Boccardi, and H. Huang, "On transceiver design and channel quantization for downlink multiuser MIMO systems with limited feedback," *IEEE Journ. Sel. Areas in Commun.*, vol. 6, no. 8, pp. 1494–1504, Oct. 2008.

[12] V. Raghavan, M. L. Honig, and V. V. Veeravalli, "Performance analysis of RVQ codebooks for limited feedback beamforming," *Proc. IEEE Intern. Symp. Inf. Theory*, pp. 2437–2441, July 2009.

[13] C.-B. Chae, D. Mazzarese, N. Jindal, and R. W. Heath, Jr., "Coordinated beamforming with limited feedback in the MIMO broadcast channel," *IEEE Journ. Sel. Areas in Commun.*, vol. 26, no. 8, pp. 1505–1515, Oct. 2008.

[14] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, NY, 4th edition, 1965.

[15] D. Hammarwall, M. Bengtsson, and B. E. Ottersten, "Acquiring partial CSI for spatially selective transmission by instantaneous channel norm feedback," *IEEE Trans. Sig. Proc.*, vol. 56, no. 3, pp. 1188–1204, Mar. 2008.

[16] D. Hammarwall, M. Bengtsson, and B. E. Ottersten, "Utilizing the spatial information provided by channel norm feedback in SDMA systems," *IEEE Trans. Sig. Proc.*, vol. 56, no. 7-2, pp. 3278–3293, July 2008.